1. Hölder space

Given a set $A \subset \mathbb{R}^n$ and a function $u : A \to \mathbb{R}$, we say that

(i) $u \in C(A)$ if u is continuous in A,

(ii) $u \in D^k(A)$ if u is k-times differentiable in A,

(iii) $u \in C^k(A)$ if u is k-times differentiable and its k-th order derivatives are continuous in A,

(iv) $u \in C^\infty(A)$ if u is smooth (∞-many times differentiable) in A,

(v) $u \in C^{0,\alpha}(A)$ if u is α-Hölder continuous in A, (see Definition below)

(vi) $u \in C^{k,\alpha}(A)$ if u is k-times differentiable and its k-th order derivatives are α-Holder continuous in A.

Definition 1. Given $\alpha \in (0, 1]$, we say u is α-Holder continuous in A is

$$[u]_{0,\alpha;A} =: \sup_{x \neq y,\, x,y \in A} \frac{|u(x) - u(y)|}{\|x - y\|^\alpha} < +\infty. \tag{1}$$

Given a multiindex $\gamma = (\gamma_1, \cdots, \gamma_k) \in \{1, \cdots, n\}^k$, we define

$$D^\gamma u = \partial_{\gamma_1} \cdots \partial_{\gamma_k} u : A \to \mathbb{R}. \tag{2}$$

For $k \in \mathbb{N}$, we define

$$[u]_{k;A} = \sup_{|\gamma| = k} \sup_A |D^\gamma u|, \tag{3}$$

and

$$\|u\|_{C^k(A)} = \sum_{m=0}^{k} [u]_{k;A}, \tag{4}$$

where

$$[u]_{0;A} = \sup_A |u|. \tag{5}$$

Also, for $\alpha \in (0, 1]$ we define

$$\|u\|_{C^\alpha(A)} = [u]_{0,\alpha;A} + \sup_A |u|. \tag{6}$$

In addition, for $k \in \mathbb{N}$ and $\alpha \in (0, 1]$, we define

$$[u]_{k,\alpha;A} = \sup_{|\gamma| = k} [D^\gamma u]_{0,\alpha;A}, \tag{7}$$

and

$$\|u\|_{C^{k,\alpha}(A)} = \|u\|_{C^k(A)} + [u]_{k,\alpha;A}. \tag{8}$$

Moreover, for $\alpha \in (0,1)$ we often denote

$$[u]_{\alpha;A} = [u]_{0,\alpha;A}, \tag{9}$$

and for $k \in \mathbb{N}$, we denote

$$[D^k u]_{\alpha;A} = \sup_{|\gamma|=k} [D^\gamma u]_{0,\alpha;A}. \tag{10}$$